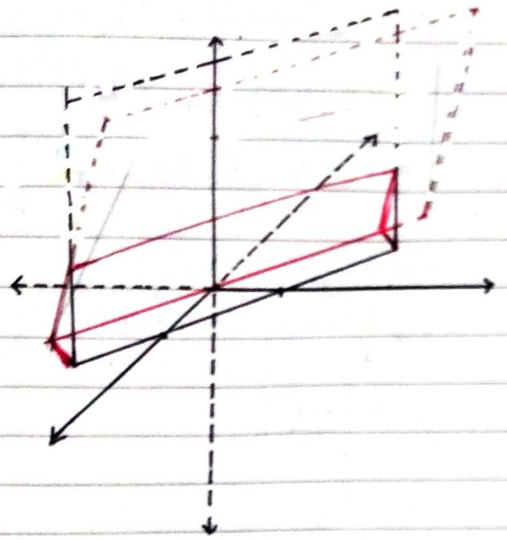
Nama : Mohammad Febryan Khamim

NRP : 5002221072

22/11/2023

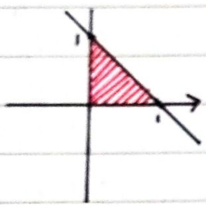
Kalkulus Peubah Banyak (B)

15. Hitung volume prisma tegak yang dibatasi : $z=0$; $x+y=1$; $x=0$; $y=0$; dan $z=x+2y$



$* \; x+y=1$

$y=1-x$



$V = \iiint_R dz\,dy\,dx$

$= \iint_A \int_0^{x+2y} dz\,dy\,dx$

$= \iint_A z \Big]_0^{x+2y} dy\,dx$

$= \int_0^1 \int_0^{1-x} (x+2y)\,dy\,dx$

$= \int_0^1 xy + y^2 \Big]_0^{1-x} dx$

$= \int_0^1 x(1-x) + (1-x)^2\,dx$

$= \int_0^1 x - x^2 + (x^2-2x+1)\,dx$

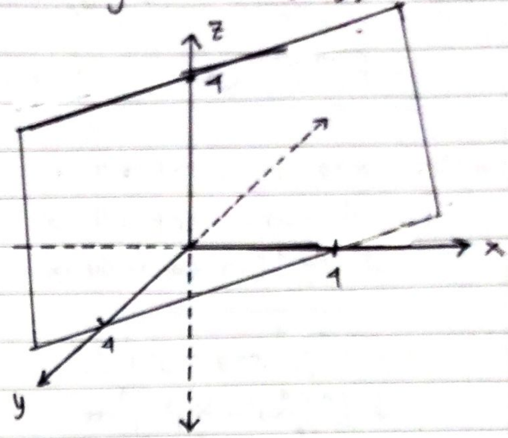
$= \int_0^1 -x+1\,dx$

$= -\frac{1}{2}x^2 + x \Big]_0^1$

$= -\frac{1}{2}+1$

$= \frac{1}{2}$ Satuan Volume //

30). Dapatkan pusat massa isi dari benda pejal yang dibatasi oleh : $x+y+z=4$ dan bidang koordinat jika $\rho = k(x+y)$



$M = \iiint_R \delta(x,y,z)\,dz\,dy\,dx$

$= \iint \int_0^{4-(x+y)} k(x+y)\,dz\,dy\,dx$

$= k \iint_A (x+y)\,z \Big]_0^{4-(x+y)} dy\,dx$

$= k \iint_A (x+y)\left[4-(x+y)\right] dy\,dx$

$= k \int_0^4 \int_0^{4-x} 4(x+y) - (x+y)^2\,dy\,dx$

misal : $u = x+y$ BA : $u=4$

$du = dy$ BB : $u = x$

$= k \int_0^4 \int_x^4 4u - u^2\,du\,dx$

$= k \int_0^4 \left[2u^2 - \frac{1}{3}u^3 \Big]_x^4 \right] dx$

$= k \int_0^4 \left[\left(32 - \frac{64}{3}\right) - \left(2x^2 - \frac{1}{3}x^3\right) \right] dx$

$= k \int_0^4 \left[\frac{32}{3} - 2x^2 + \frac{1}{3}x^3 \right] dx$

$= k \left[\frac{32}{3}x - \frac{2}{3}x^3 + \frac{1}{12}x^4 \Big]_0^4 \right]$

$= k \left[\frac{128}{3} - \frac{128}{3} + \frac{256}{12} \right]$

$= k \left[\frac{64}{3} \right]$

$= 21\frac{1}{3}k$

$$M_{yoz} = \iiint_R x\ \delta(x,y,z)\, dz\,dy\,dx$$
$$= \iint\int_0^{4-(x+y)} x\,(k(x+y))\,dz\,dy\,dx$$
$$= k\int_0^4\int_0^{4-x} (x^2+xy)\,z\Big]_0^{4-(x+y)} dy\,dx$$
$$= k\int_0^4\int_0^{4-x} (x(x+y))(4-(x+y))\,dy\,dx$$

misal : $x+y=u$ BA $: u=4$

$du = dy$ BB $: u = x$

$$= k\int_0^4\int_x^4 (xu)(4-u)\,du\,dx$$
$$= k\int_0^4\int_x^4 4xu - xu^2\,du\,dx$$
$$= k\int_0^4 2xu^2 - \frac{x}{3}u^3\Big]_x^4 dx$$
$$= k\int_0^4 \left(32x - \frac{64x}{3}\right) - \left[2x^3 - \frac{x^4}{3}\right] dx$$
$$= k\int_0^4 \frac{32}{3}x - 2x^3 + \frac{x^4}{3}\,dx$$
$$= k\left[\frac{16}{3}x^2 - \frac{1}{2}x^4 + \frac{x^5}{15}\right]_0^4$$
$$= k\left[\frac{256}{3} - 128 + \frac{1024}{15}\right]$$
$$= k\left[\frac{1280}{15} - \frac{1920}{15} + \frac{1024}{15}\right]$$
$$= k\left[\frac{384}{15}\right]$$
$$= \frac{128}{5}k$$

$$M_{xoz} = \iiint_R y\ \delta(x,y,z)\,dz\,dy\,dx$$
$$= \iiint_0^{4-(x+y)} y\,(k(x+y))\,dz\,dy\,dx$$
$$= \iint k\,[y(x+y)]\,z\Big]_0^{4-(x+y)} dy\,dx$$
$$= k\iint_A [y(x+y)][4-(x+y)]\,dy\,dx$$

misal : $u = x+y$ BA $: u = 4$

$du = dx$ BB $: u =$

$$= k\int_0^4\int_y^4 [uy][4-u]\,du\,dy$$
$$= k\int_0^4\int_y^4 4uy - u^2y\,du\,dy$$
$$= k\int_0^4 \left[2u^2y - \frac{y}{3}u^3\Big]_y^4\right] dy$$
$$= k\int_0^4 \left[\left(32y - \frac{64}{3}y\right) - \left(2y^3 - \frac{y^4}{3}\right)\right] dy$$
$$= k\int_0^4 \left[\frac{32}{3}y - 2y^3 + \frac{y^4}{3}\right] dy$$
$$= k\left[\frac{32}{6}y^2 - \frac{1}{2}y^4 + \frac{y^5}{15}\Big]_0^4\right]$$
$$= k\left[\frac{32.16}{6} - \frac{1}{2}.256 + \frac{1024}{15}\right]$$
$$= k\left[\frac{256}{3} - 128 + \frac{1024}{15}\right]$$
$$= k\left[\frac{1280 - 1920 + 1024}{15}\right]$$
$$= k\left[\frac{384}{15}\right]$$
$$= \frac{128}{5}k$$

$$M_{xoy} = \iiint_R z\ \delta(x,y,z)\,dz\,dy\,dx$$
$$= \iint\int_0^{4-(x+y)} z\,(k(x+y))\,dz\,dy\,dx$$
$$= k\iint \frac{1}{2}z^2(x+y)\Big]_0^{4-(x+y)} dy\,dx$$
$$= \frac{k}{2}\iint [4-(x+y)]^2(x+y)\,dy\,dx$$

misal : $x+y=u$ BA $: u = 4$

$du = dy$ BB $: u = x$

$$= \frac{k}{2}\iint [4-u]^2 u\,dy\,dx$$
$$= \frac{k}{2}\int_0^4\int_x^4 [16 - 8u + u^2]u\,dy\,dx$$
$$= \frac{k}{2}\int_0^4\int_x^4 16u - 8u^2 + u^3\,dy\,dx$$
$$= \frac{k}{2}\int_0^4 8u^2 - \frac{8}{3}u^3 + \frac{1}{4}u^4\Big]_x^4 dx$$
$$= \frac{k}{2}\int_0^4 \left[\left(128 - \frac{512}{3} + 64\right) - \left(8x^2 - \frac{8}{3}x^3 + \frac{x^4}{4}\right)\right] dx$$
$$= \frac{k}{2}\int_0^4 \left[\frac{384 - 512 + 192}{3}\right] - \left(8x^2 - \frac{8}{3}x^3 + \frac{x^4}{4}\right) dx$$
$$= \frac{k}{2}\int_0^4 \left[\frac{64}{3} - 8x^2 + \frac{8}{3}x^3 - \frac{x^4}{4}\right] dx$$
$$= \frac{k}{2}\left[\frac{64}{3}x - \frac{8}{3}x^3 + \frac{2}{3}x^4 - \frac{x^5}{20}\right]_0^4$$
$$= \frac{k}{2}\left[\frac{256}{3} - \frac{512}{3} + \frac{512}{3} - \frac{1024}{20}\right]$$
$$= \frac{k}{2}\left[\frac{256}{3} - \frac{256}{5}\right]$$
$$= \frac{k}{2}\left[\frac{256(5-3)}{15}\right]$$
$$= \frac{k}{2}\left[\frac{512}{15}\right]$$

SiDU

$$M_{xoy} = \frac{512}{30} k$$

$$\bar{x} = \frac{M_{yoz}}{M} = \frac{128/5\ \cancel{k}}{64/3\ \cancel{k}}$$

$$= \frac{\overset{2}{\cancel{128}}}{5} \cdot \frac{3}{\cancel{64}}$$

$$= \frac{6}{5}$$

$$= 1{,}2$$

$$\bar{y} = \frac{M_{xoz}}{M} = \frac{128/5\ \cancel{k}}{64/3\ \cancel{k}}$$

$$= \frac{\overset{2}{\cancel{128}}}{5} \cdot \frac{3}{\cancel{64}}$$

$$= \frac{6}{5}$$

$$= 1{,}2$$

$$\bar{z} = \frac{M_{xoy}}{M} = \frac{512/30\ \cancel{k}}{64/3\ \cancel{k}}$$

$$= \frac{512}{\underset{10}{\cancel{30}}} \cdot \frac{\cancel{3}}{64}$$

$$= \frac{8}{10}$$

$$= 0{,}8$$

Pusat Massa $= (\bar{x}, \bar{y}, \bar{z})$

$= (1.2,\ 1.2,\ 0.8)$

31) $I = \iiint_R \zeta(x,y,z)\, dx\, dy\, dz$, density $= \zeta(x,y,z)$

$= x^2+y^2+z^2$

a). Sketsakan daerah Integrasi I berikut.

$$\int_0^{3\sqrt{2}} \int_0^{\sqrt{18-y^2}} \int_0^{\sqrt{18-(x^2+y^2)}} \zeta(x,y,z)\, dz\, dy\, dx$$

$$= \int_0^{3\sqrt{2}} \int_0^{\sqrt{18-y^2}} \int_0^{\sqrt{18-(x^2+y^2)}} (x^2+y^2+z^2)\, dz\, dy\, dx$$

•> Pada ruang :

$z = \sqrt{18-(x^2+y^2)}$

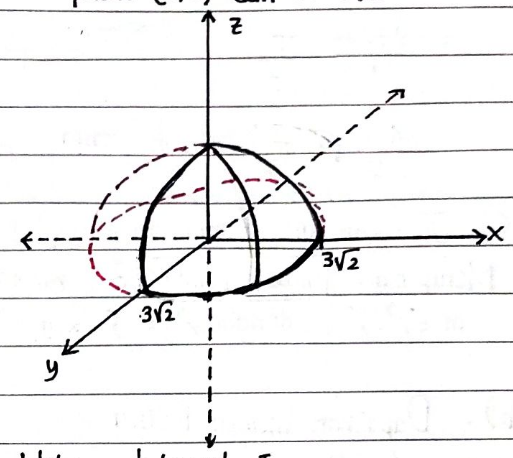
$z^2 = 18 - (x^2+y^2)$

$x^2+y^2+z^2 = 18$ (bola di z positif) $\rightarrow r = 3\sqrt{2}$ dan pusat $(0,0,0)$

•> Di bidang xy

$x = \sqrt{18 - y^2}$

$x^2 + y^2 = (3\sqrt{2})^2 \rightarrow$ lingkaran di x positif. pusat $(0,0)$ dan $r = 3\sqrt{2}$



b). Hitung Integral I

$$\int_0^{3\sqrt{2}} \int_0^{\sqrt{18-y^2}} \int_0^{\sqrt{18-(x^2+y^2)}} (x^2+y^2+z^2)\, dz\, dy\, dx$$

※ Ubah ke Koordinat Bola

$x = \rho \sin\theta \cos\varphi$ $\quad y = \rho \sin\theta \sin\varphi$

$z = \rho \cos\theta$ $\quad |J| = \rho^2 \sin\theta$

•> $z = \sqrt{18-(x^2+y^2)}$

$\rho\cos\theta = \sqrt{18 - (\rho^2 \sin^2\theta \cos^2\varphi + \rho^2 \sin^2\theta \sin^2\varphi)}$

$\rho^2\cos^2\theta = 18 - (\rho^2 \sin^2\theta\, [\cos^2\varphi + \sin^2\varphi])$

$\rho^2\cos^2\theta = 18 - (\rho^2\sin^2\theta)$

$\rho^2\cos^2\theta + \rho^2\sin^2\theta = 18$

$\rho^2 = 18$

$\rho = 3\sqrt{2} \rightarrow$ dari 0 sampai $3\sqrt{2}$

•> $x^2+y^2+z^2 = \rho^2\cos^2\theta + \rho^2\sin^2\theta\,(\cos^2\varphi + \sin^2\theta]$

$= \rho^2(\cos^2\theta + \sin^2\theta)$

$= \rho^2$

$$\Rightarrow \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3\sqrt{2}} \rho^2 \cdot \rho^2 \sin\theta\, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{3\sqrt{2}} \rho^4 \sin\theta\, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \left.\frac{1}{5}\rho^5\right]_0^{3\sqrt{2}} \sin\theta\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} \frac{972\sqrt{2}}{5} \sin\theta\, d\theta\, d\varphi$$

$$= \frac{972\sqrt{2}}{5} \int_0^{\pi/2} \left. -\cos\theta \right]_0^{\pi/2} d\varphi$$

$$= \frac{972\sqrt{2}}{5} \int_0^{\pi/2} (-0+1)\, d\varphi$$

$$= \frac{972\sqrt{2}}{5} \int_0^{\pi/2} d\varphi$$

$$= \frac{972\sqrt{2}}{5} \cdot \varphi \Big]_0^{\pi/2}$$

$$= \frac{972\sqrt{2}}{5} \cdot \frac{\pi}{2}$$

$$= \frac{486}{5}\sqrt{2}\,.\,\pi \text{ satuan massa}$$

c). Jelaskan arti interpretasi integral I tersebut

Merupakan $\boxed{\text{massa benda}}$ R, karena

$m = \rho . V$, dengan $\rho = \zeta(x,y,z)$

32. b). Dapatkan momen inersia

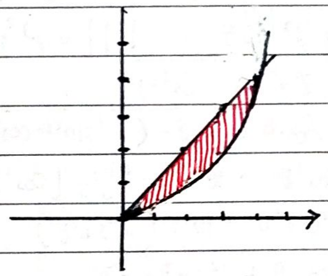
$4y = x^2$ dan $y = x$ terhadap y

• $y = \frac{1}{4}x^2$ dan $y = x$

$x = 1 \rightarrow y = \frac{1}{4}$

$x = 2 \rightarrow y = 1$

$x = 4 \rightarrow y = 4$



$$I_y = \int_0^4 \int_{\frac{1}{4}x^2}^{x} x^2\,dy\,dx = \int_0^4 x^2 y \Big]_{\frac{1}{4}x^2}^{x} dx$$

$$= \int_0^4 x^3 - \frac{1}{4}x^4\,dx$$

$$= \frac{x^4}{4} - \frac{x^5}{20} \Big]_0^4$$

$$= \frac{5(4)^4 - 4^5}{20}$$

$$= \frac{4^4(5-4)}{4.5}$$

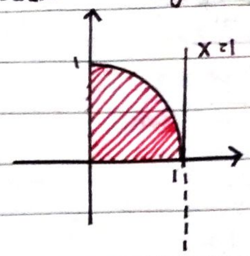
$$= \frac{4^3}{5}$$

$$= \frac{64}{5}$$

$$= 12\tfrac{4}{5} \text{ satuan inersia}$$

34). c). Dapatkan massa keping datar D yang dibatasi kurva : $x^2 + y^2 = 1$, $x = 1$ di Kuadran 1 dengan $\delta(x,y) = xy$

I



$$M = \iint_A \delta(x,y)\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy\,dy\,dx$$

$$= \int_0^1 \frac{x}{2}y^2 \Big]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{x(1-x^2)}{2}\,dx$$

$$= \frac{1}{2}\int_0^1 x - x^3\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\Big]_0^1\right]$$

$$= \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{1}{2}\left(\frac{1}{4}\right)$$

$$= \frac{1}{8} \text{ satuan volume}$$

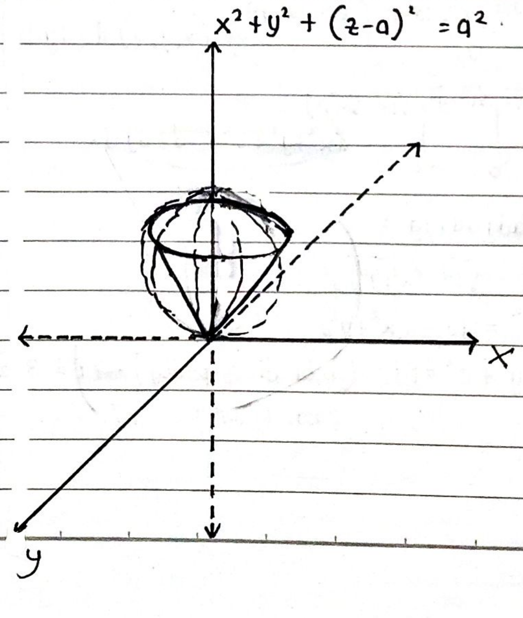
36). Isi benda di atas kerucut $z^2 = x^2 + y^2$ dan dalam bola $x^2 + y^2 + z^2 = 2az$

•> Bola : $x^2 + y^2 + z^2 = 2az$

$$x^2 + y^2 + z^2 - 2az = 0 \quad + a^2$$

$$x^2 + y^2 + (z^2 - 2az + a^2) = a^2$$

$$x^2 + y^2 + (z-a)^2 = a^2$$



36. Batas Integral :

- Luar : $x^2+y^2+z^2 = 2az$

$$\rho^2 = \rho\cos\theta \cdot 2a$$
$$\rho = 2a\cos\theta$$

- Dalam : $z^2 \; x^2 + y^2$

$$\rho^2\cos^2\theta = \rho^2\sin^2\theta$$
$$\rho^2(\cos^2\theta - \sin^2\theta) = 0$$
$$\rho^2 = 0$$
$$\rho = 0$$

$\cdot\!> 0 \le \varphi \le 2\pi$ $\quad \cdot\!> 0 \le \theta \le \pi/2$ $\quad \cdot\!> 0 \le \rho \le 2a\cos\theta$

$$|J| = \rho^2\sin\theta$$

$$= \iiint_R dy\,dx\,dz$$

$$= \iiint_R |J|\,d\rho\,d\theta\,d\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{2a\cos\theta} \rho^2\sin\theta\,d\rho\,d\theta\,d\varphi$$

$$= 4\int_{\varphi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} \frac{1}{3}\rho^3\Big]_0^{2a\cos\theta} \sin\theta\,d\theta\,d\varphi$$

$$= \frac{4}{3}\int_{\varphi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} 8a^3\cos^2\theta\,\sin\theta\cos\theta\,d\theta\,d\varphi$$

$$= \frac{4}{3}\cdot 4a^3\int_{\varphi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} 2\sin\theta\cos\theta\cdot\cos^2\theta\,d\theta\,d\varphi$$

misal $= \cos\theta = u$ $\quad d\theta = -\dfrac{dk}{\sin\theta}$

$dk = -\sin\theta\,d\theta$

BA : $k = \cos\pi/2$, $k = 0$ $\qquad$ BB : $k = \cos 0$, $k = 1$

$$\int_{\theta=0}^{\pi/2} 2\cancel{\sin\theta}\cos^3\theta \cdot -\frac{dk}{\cancel{\sin\theta}}$$

$$= -2\int_{k=1}^{k=0} k^3\,dk$$

$$= -2\left[\frac{1}{4}k^4\right]_1^0$$

$$= \frac{-2}{4}\left[0-1\right]$$

$$= \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow \frac{4}{3}\cdot 4a^3\int_{\varphi=0}^{\pi/2}\frac{1}{2}\,d\varphi$$

$$= \frac{8}{3}a^3\left[\varphi\right]_0^{\pi/2}$$

$$= \frac{\overset{4}{\cancel{8}}}{3}a^3\cdot\frac{\pi}{\cancel{2}}$$

$$= \frac{4}{3}\pi a^3 \text{ satuan Volume}$$